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Finite Groups Acting on Surfaces and the Genus of a Group

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The relationship between a finite group action on a closed surface and Cayley graphs for the group embedded in the surface is studied.

The genus $\gamma(A)$ of the group A is the smallest integer g such that some Cayley graph for A can be embedded in the surface of genus g. The modern study of the genus of a group, initiated by White [28], was a natural outgrowth of the use of Cayley graphs in the proof of the Heawood map color theorem by Ringel, Youngs, *et al.* The genera of "most" Abelian groups have been computed by Jungerman and White [11], but little was know about non-Abelian groups until recently when Proulx [20, 21] classified groups of genus one. (The classification of groups of genus zero was done by Maschke at the turn of the century [18].) The author then used Proulx's results to bound the order of any group having a given genus greater than one [26]. Strangely enough, this bound turned out to be twice the Hurwitz $84(g-1)$ bound for the order of the group of conformal automorphisms of a Riemann surface of genus $g > 1$. When we began work on [26] we were not even aware of Hurwitz's theorem, but we would not have expected an analog for Cayley graphs in any case. The purpose of this paper is to explore further the relationship between embedded Cayley graphs and group actions on surfaces.

It should not be surprising, considering how long and how much this field has been studied, that some results of this paper, such as the group presentations given by Theorem 4 or the detailed Hurwitz theorem of Section 6, are already well known in one form or another. Some aspects of this paper are unusual, however. The machinery of Riemann surfaces, uniformization, and conformal automorphisms is replaced by the combinatorial viewpoint of branched coverings given by the voltage graph construction. Also, Cayley

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graph embeddings for groups generated by involutions force us to consider group actions with reflections; such actions lead to significant complications and consequently have received less attention. In any case, even the traditional material of this paper may be unfamiliar to many combinatorists and topologists (witness our original ignorance of Hurwitz's theorem), and sources are sometimes obscure (we thank a referee for supplying additional references).

Section 1 of this paper contains background material on group actions, branched coverings, and surface homeomorphisms of finite order. Proposition 1 does not seem to be well known amongst topologists, at least in the given generality. The main point of Section 2 was originally to show that if a group acts on an orientable surface, then a Cayley graph for the group embeds in the surface. In the process, we also obtain partial presentations for a group given geometric information about its action on a surface. Section 3 introduces the symmetric genus of a group, the smallest genus of any surface on which the group acts. It is shown that genus and symmetric genus can be quite different for Abelian groups. Section 4 gives a viewpoint of the classification of groups of genus zero or one in terms of spherical and Euclidean space groups. Section 5 gives a refined version of the Hurwitz $84(g-1)$ theorem that includes orientation reversing group actions. Finally, Section 6 studies some of the problems arising from group actions on nonorientable surfaces.

1. Finite Group Actions on Closed Surfaces

Throughout this paper, a *surface* is a connected compact 2-manifold without boundary; that is, all surfaces here are closed. A *graph* is a finite 1-dimensional CW complex; thus a graph may have loops and multiple edges. The identity element of a group is denoted 1. The number of elements in a set X is denoted $|X|$. An embedding of a graph G in a surface S is *cellular* if each component ("face") of the complement $S-G$ is homeomorphic to the interior of the unit disk.

An *action of a group A on a topological space X* is given by an isomorphism of the group A onto a subgroup of the group of all homeomorphisms of X. We will not differentiate between the abstract group A and the subgroup of homeomorphisms; if $a \in A$ we will refer to the "homeomorphism a." The *stabilizer* of $x \in X$ is $A_x = \{a \in A \mid a(x) = x\}$ and $\operatorname{Fix} A = \{x \in X \mid A_x \neq 1\}$. An action of A on X is *free* if $\operatorname{Fix} A$ is empty and *pseudo-free* if $\operatorname{Fix} A$ is nonempty but discrete. In all our applications X is a surface or a graph and A is a finite group. In this case, $\operatorname{Fix} A$ must be finite if the action is pseudo-free.

Given an action of the group A on the space X, the *orbit* of a point $x \in X$ is the set $[x] = \{y \mid y = a(x) \text{ for some } a \in A\}$. The quotient space X/A is the set of all orbits with the topology that $U \subset X/A$ is open if and only if $p^{-1}(U)$ is open, where $p: X \to X/A$ is the *natural projection* $p(x) = [x]$.

Suppose that A is a finite group acting pseudo-freely on the surface S. Then S/A is a surface and the natural projection $p: S \to S/A$ is a *branched covering*, that is, a local homeomorphism except at points in $\operatorname{Fix} A$. The set $p(\operatorname{Fix} A)$ is called the *branch* set. If $x \in \operatorname{Fix} A$, then p is locally $|A_x|$-to-1 in a neighborhood of x. If $y \in p(\operatorname{Fix} A)$, then $|A_x|$ is the same for any $x \in p^{-1}(y)$. The common number is called the *order* of the branch point y and denoted m_y. It follows that $|p^{-1}(y)| = |A|/m_y$.

The Euler characteristic for S is easily computed from the Euler characteristic for S/A and the orders of the branch points. Simply triangulate S/A so that every branch point is a vertex of the triangulation T. Then $p^{-1}(T)$ is a triangulation of S. Since $|p^{-1}(t)| = |A|$ if t is an edge, face, or nonbranch point vertex of T,

$$\chi(S) = |A| \left(\chi(S/A) - \sum_y (1 - 1/m_y)\right),$$

where the sum is taken over the branch set. This equation is called the *Riemann–Hurwitz equation.*

We would like to know when a finite group acts pseudo-freely on a surface. The following classification theorem for homeomorphisms of finite order is crucial.

PROPOSITION 1. *Let S be an orientable surface and $h: S \to S$ a homeomorphism of finite order. Then one of the following must occur*:

(a) *h has a finite number of fixed points,*

(b) *h is an orientation reversing involution and $S = S_1 \cup S_2$, where S_1 is connected, $h(S_1) = S_2$, and $S_1 \cap S_2$ is a finite collection of disjoint simple closed curves, at least one of which is left point-wise fixed by h.*

The proof of Proposition 1 is straightforward if h is a conformal automorphism or a linear map with respect to some triangulation of S. It is almost as easy if h is piecewise linear. However, without such conditions the proof is long and difficult (see Kerékjártó [12, 13] and Scherrer [24]). A homeomorphism of type (b) will be called a *reflection*, components of $S_1 \cap S_2$ *fixed circles*, and S_1, S_2 *halves* of the reflection. It follows from Proposition 1 that any finite group acting on an orientable surface S acts pseudo-freely unless the action has a reflection. In particular, if A^o denotes the subgroup of orientation preserving homeomorphisms of an action of A on an orientation surface, then A^o always acts pseudo-freely. As A^o has index at most two in A, it is not hard in many situations to get around reflections by passing to A^o.

PROPOSITION 2. *Let A be a finite group acting on the orientable surface S. Let B be a normal subgroup of A. Then A/B acts on S/B so that $(S/B)/(A/B) = S/A$. If B has no reflections, then S/B is a surface, orientable if $B \subset A^o$ and nonorientable otherwise. In any case, $S/B \cap A^o$ is an orientable surface and A/B acts on it if $B \not\subset A^o$.*

Proof. Let $a \to \bar{a}$ denote the natural homeomorphism of A onto A/B. The desired action of A/B on S/B is given by defining $\bar{a}([x]) = [a(x)]$. It is an exercise in permutation groups and general topology to verify that this definition does not depend on the choice of representatives for $\bar{a}$ or $[x]$, that $\bar{a}$ is in fact a homeomorphism of S/B, and that $(S/B)/(A/B)$ is homeomorphic to S/A in a natural way. If B has no reflections, the action of B is pseudo-free so that S/B is a surface. Finally, by the arguments already given, $S/A^o \cap B$ is an orientable surface and $A^o/A^o \cap B$ acts on it. If $B \not\subset A^o$, then $BA^o = A$ since A^o has index 2 in A. Therefore $A^o/A^o \cap B$ is isomorphic to $BA^o/B = A/B$. ∎

2. Cayley Graph Imbeddings

Let A be a group and X a generating set for A. The *Cayley* (color) *graph* for A and X, denoted $C(A, X)$, is the graph having vertex set A and an edge from a to ax ("colored" x) for each $a \in A$ and $x \in X$. There is some variation as to whether or not the pair of edges between a and ax, corresponding to an involution $x \in X$, should be identified to one edge. Either is acceptable here as long as the policy is consistent for all edges of the same color. The following proposition, often assumed without comment, is proved by Sabidussi in [22].

PROPOSITION 3. *The graph G is a Cayley graph $C(A, X)$ for the group A if and only if A acts on G transitively on vertices so that no vertex of G is left fixed by a nonidentity element of A. Edges in the same orbit of A correspond to the same generator in X.*

Notice that an element $a \in A$ may have fixed points in the interior of an edge; this happens when a is an involution interchanging the endpoints of an edge and depends upon the identification to a single edge of a pair of edges corresponding to an involution.

Let G be a graph cellularly embedded in the surface S and let α be a map from the set E of directed edges of G to a group A satisfying $\alpha(e^-) = \alpha(e)^{-1}$

for each $e \in E$, where e^- denotes the reverse direction of e. Then the voltage graph construction [7, 8] yields a derived surface S^α on which A acts pseudo-freely with quotient S. The important facts needed here about this construction are as follows: The natural projection of the action $p: S^\alpha \to S = S^\alpha/A$ has no branch points lying on G and at most one branch point in the interior of any face. The order of the branch point in the face f is the order of the product $\alpha(e_1) \cdots \alpha(e_n)$, where the directed edges $e_1,..., e_n$ give the directed path in G forming the boundary of f; in particular, if f contains no branch point $\alpha(e_1) \cdots \alpha(e_n) = 1$. By the results of [7] every pseudo-free group action on a surface can be obtained from a voltage graph construction on any graph cellularly embedded in the quotient surface, missing the branch points. If the graph G has one vertex, then the derived surface is connected if and only if $\{\alpha(e), e \in E\}$ generate the group A. If the surface S is nonorientable and G has one vertex, the derived surface S^α is orientable if and only if there is an index 2 subgroup A^o of the group A such that $\alpha(e) \in A^o$ when the edge e is an orientation preserving loop on the surface and $\alpha(e) \notin A^o$ when e is an orientation reversing loop (see [8]).

We will show that if a group acts on an orientable surface, then a Cayley graph for the group embeds in the surface. Along the way we will also give necessary and sufficient conditions that are satisfied by some presentation of the group in order that it act on the given surface in a given fashion. The partial presentations we give are fairly complicated. To quell any uprisings from the many subscripts needed, we introduce some notational conveniences. In listing generators for a presentation, we indicate the range of a subscript in the subscript; thus "a_g" means "$a_1, a_2,..., a_g$." In giving relations, we only list a representative relation for each subscript; for example, if y_t is given in a generating set, then $y^m = 1$ means $y_1^{m_1} = \cdots = y_t^{m_t} = 1$, or if ${}_r z_{s(r)+1}$ is given in the generating set, then $(z_j z_{j+1})^{q_j}$ means $({}_i z_j)({}_i z_{j+1})^{q_j} = 1$ for all $j = 1,..., s(i) + 1$ and all $i = 1,..., r$. Finally, $\Pi y = y_1 y_2 \cdots y_t$ and $[a, b] = aba^{-1}b^{-1}$.

THEOREM 4. *Let A be a finite group acting without reflections on the orientable surface S. Let $n = 2 - \chi(S/A) = 2 - 2g$ and let $p: S \to S/A$ have t branch points of order $m_1,..., m_t$. Then there is a Cayley graph $C(A, X)$ cellularly embedded in S, where X is the generating set in one of the following partial presentations for A:*

(a) $\langle a_g, b_g, y_t: y^m = 1, \Pi[a, b]\, \Pi y = 1,... \rangle$ *if* $A = A^o$,

(b) $\langle c_n, y_t: y^m = 1, \Pi c^2 \Pi y = 1,... \rangle$ *if* $A \neq A^o$.

Moreover, in case (b), *A^o contains every y but no c.*

Proof. Observe that S/A is in fact a surface and $p: S \to S/A$ is a branched covering since A acts without reflections. The usual representation

[19] of the surface of Euler characteristic $2-n$ as a $2n$-sided polygon with sides to be identified in pairs gives an embedding in the surface S/A of a one vertex graph having n directed loops $d_1,..., d_n$. If S/A is orientable ($A=A^o$) the embedding has one face bounded by the circuit $d_1 d_2 d_1^- d_2^- \cdots d_n^-$. If S/A is nonorientable ($A \neq A^o$) the single face of the embedding is bounded by the circuit $d_1 d_1 \cdots d_n d_n$. This embedded graph clearly can be chosen to avoid the branch points. Now add one loop around each branch point, t loops in all, $e_1,..., e_t$. Call the resulting one vertex graph G. Then $p^{-1}(G)$ is the desired Cayley graph: if S/A is orientable a_i corresponds to d_{2i-1} and b_i to d_{2i}, and if S/A is nonorientable c_i corresponds to d_i. The relations in presentations (a) and (b) follow directly from the facts already given about the voltage graph construction. ∎

Presentation (a), as a quotient of a Fuchsian group, is well known. Presentation (b) is less known but can be found in Zieschang, Vogt, and Coldeway [29] or Macbeath [15].

The situation is somewhat more complicated when A contains reflections. Then A/A^o acts on the surface $S^o = S/A^o$ as a single reflection h. We seek a 2-vertex graph G cellularly embedded in S^o so that every face contains at most one branch point and such that the reflection leaves G invariant ($h(G)=G$) but interchanges the two vertices of G. The idea is to construct half, G', of G and then reflect this half to get the other half of G. If S_1^o respresents half of the surface S^o under the reflection, then G' must take into account the genus g of S_1^o, the boundary components $C_1,..., C_n$ of S_1^o (the fixed circles of h), the branch points of A^o lying inside S_1^o, and the branch points of A^o lying on the fixed circles of h.

The construction of G' follows. Choose a vertex v in the interior of S_1^o. Add $2g$ loops $d_1,..., d_{2g}$ based at v so that their complement in S_1^o is homeomorphic to an disk with r holes corresponding to the r boundary components $C_1,..., C_r$. Run disjoint loops $e_1,..., e_t$ around the t branch points of A^o in the interior of S_1^o. Also run r disjoint loops $c_1,..., c_r$ around the boundary components $C_1,..., C_r$ so that c_i and C_i bound an annulus containing no other edge of the graph constructed so far. Finally, suppose A^o has $s(i)$ branch points on the circle C_i. Let ${}_iu_j$, $1 \leqslant j \leqslant s(i)+1$, be points in cyclic order around C_i such that the circular arc from ${}_iu_j$ to ${}_iu_{j+1}$ contains exactly one branch point, $1 \leqslant j \leqslant s(i)$. Notice that the circular arc from ${}_iu_{s(i)+1}$ to ${}_iu_1$ contains no branch points. Run disjoint paths ${}_if_j$ from v to ${}_iu_j$ for all $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s(i+1)$. Call the resulting collection of loops and paths G'. Figure 1 illustrates G' using two "overlays" for the case that $g=1$, $r=2$, $t=1$, $s(1)=0$, $s(2)=1$ (interiors of all edges are disjoint).

Let $G = G' \cup h(G')$, where the points ${}_iu_j$ are not counted as vertices. By construction, A acts on $p^{-1}(G)$ leaving no vertex fixed. Each loop in G' corresponds to a generator of A contained in A^o. Each half-edge ${}_if_j$ in G'

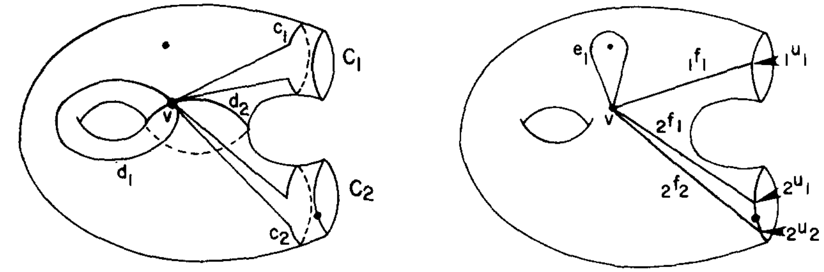

FIG. 1. The graph G' in S_1^o. Dots represent branch points.

corresponds to a reflection element of A. Relations amongst the generators can be read off from the faces of the embedding using the voltage graph construction. To summarize:

THEOREM 5. *Let A act on the orientable surface S with reflections. Then A/A^o acts on S/A^o as a single reflection. Let $C_1, \ldots, C_r$ be the fixed circles of the reflection and S_1^o a half of the reflection. Let A^o have t branch points in the interior of S_1^o of orders $m_1, \ldots, m_t$, and $s(i)$ branch points of order ${}_iq_j$, $1 \leqslant j \leqslant s(i)$, on the circle C_i, $1 \leqslant i \leqslant r$. Then there is a Cayley graph $C(A, X)$ for A cellularly embedded in S, where X is the generating set for the partial presentation:*

(c) $\langle a_g,\ b_g,\ y_t,\ w_r,\ {}_rz_{s(r)+1}:\ y^m = 1,\quad z^2 = 1,\quad (z_j z_{j+1})^{q_j} = 1,$ $wz_1w^{-1}z_{s(i)+1} = 1,\ \Pi[a, b]\ \Pi y\ \Pi w = 1, \ldots\rangle$

Moreover, A^o contains every a, b, y, and w but no z.

Observe that in the case that S/A^o is the sphere, there are no a's or b's and since there can be only one fixed circle, the z's need not be doubly subscripted. The partial presentation is then not so unwieldy. Presentation (c) is a quotient of a non-Euclidean space group (see Section 4) and appears in [15, 29]. In particular, Fig. 1 may be viewed as a marked fundamental region for the associated space group.

It should also be noted that the Cayley graph embedding obtained in each of the Theorems 4 and 5 is invariant under the action of the group A on the surface S.

There are converses for Theorems 4 and 5 as well. For the most part, the proofs follow directly from the voltage graph construction.

THEOREM 6. *Let A be a finite group having a partial presentation of type* (a) *or* (b). *Then A acts pseudo-freely on a surface S of Euler characteristic $\chi(S) = |A|(2 - n - \sum_{k=1}^{t}(1 - 1/m_k))$. If the presentation is type* (a), *then S is orientable and the action of A is orientation preserving. If the* *presentation is type* (b) *and there is a subgroup of index* 2 *in* A *containing every* y *but no* c, *then* S *is orientable but the action is not orientation preserving. Otherwise,* S *is nonorientable.*

Proof. Embed a bouquet of circles in a surface S^o of Euler characteristic $2-n$ as in the proof of Theorem 4. The surface S^o is orientable for type (a) and nonorientable for type (b). Assign voltages again using the proof of Theorem 4. Then the desired surface S is the surface derived from the voltage graph construction. The orientability of S follows from previous remarks on this construction, and its Euler characteristic is given by the Riemann–Hurwitz equation. ∎

THEOREM 7. *Let* A *be a finite group having a partial presentation of the form* (c) *from Theorem* 5, *such that there is a subgroup* B *of index* 2 *in* A *containing every* a, b, y, *and* w *but no* z. *Then* A *acts with reflections on the orientable surface* S *of Euler characteristic*

$$\chi(S)=|A|\left(2-2g-r-\sum_k(1-1/m_k)-\sum_{i,j}(1-1/_iq_j)/2\right).$$

Proof. The orientable surface S^o such that $\chi(S^o)=2(2-2g-r)$ can be split by r simple closed curves into two connected surfaces S_1^o, S_2^o each having genus g and r boundary components. Embed a two-vertex graph G in S^o as in the discussion for Theorem 5. Assign voltages to G in the group A again using the proof of Theorem 5 for instructions. It can be shown using [8] that the derived surface has two components and that B acts pseudo-freely on each. Let S be one of the components. The reflection of S^o with halves S_1^o and S_2^o then lifts to a reflection of S that together with B generates the desired action of A on S. Observe that, by construction, for each branch point of B in the interior of S_1^o there is a reflected branch point of the same order in S_2^o. Therefore by the Riemann–Hurwitz equation

$$\chi(S)=|B|\left(\chi(S^o)-2\sum(1-1/m_k)-\sum(1-1/_iq_j)\right).$$

Use $|B|=|A|/2$ and $\chi(S^o)=2(2-2g-r)$ to obtain the desired equation for $\chi(S)$. ∎

3. Symmetric Genus

Define the *symmetric genus* $\sigma(A)$ of a finite group A to be the minimum genus of any surface on which A acts. Let the *strong symmetric genus* $\sigma^o(A)$ be the minimum genus of any surface on which A acts preserving orientation. Obviously $\sigma(A)\leqslant\sigma^o(A)$. Theorems 4 and 5 imply

COROLLARY 8. $\gamma(A) \leqslant \sigma(A)$.

The strong symmetric genus, in one guise or another, has been studied by Burnside [3] (with the restriction that the quotient surface be the sphere), Maclachlin [16] (with the restriction that the surface on which A acts has genus greater than one), and Levinson and Maskit [14] (where it is called the point symmetric genus). On the other hand, the symmetric genus seems not to have been studied. Levinson and Maskit do define the weak point symmetric genus $w(A)$ to be the minimum genus of a surface in which a Cayley graph for A can be embedded so that the rotation at every vertex is the same or reversed (the rotation at a vertex is the cyclic ordering of the edges incident to the vertex given by an orientation of the surface). It follows from the proof of Theorem 5 that $w(A) \leqslant \sigma(A)$. However, weakly point symmetric embeddings of Cayley graphs in general have nothing to do with group actions involving orientation reversing elements. Since any assignment of vertex rotations defines an embedding in an orientable surface [6], every Cayley graph has weakly point symmetric embeddings. Indeed, every embedding of a trivalent Cayley graph is weakly point symmetric. However, if the group for the Cayley graph has no index 2 subgroups, for example if its order is odd, then no action of that group can have an orientation reversing element. Levinson and Maskit may not have realized this, for [14, Theorem 6] is clearly false.

One would expect $\gamma(A) < \sigma(A)$ in general. Since $\gamma(A)$ and $\sigma(A)$ are known for so few groups, however, this is hard to verify. Maclachlin 16] has computed $\sigma^o(A)$ for all Abelian groups and of course $\sigma^o(A) = \sigma(A)$ if A has odd order. Jungerman and White [11] have computed the genus of "most" Abelian groups. In all these cases, if $\gamma(A) > 1$, then $\gamma(A) < \sigma^o(A)$; in fact $\sigma^o(A)$ is around $2\gamma(A)$. This suggests that the problem lies in not allowing orientation reversing homeomorphisms, at least for even order groups. However, a careful inspection of Theorem 5 and Maclachlin's work will likely show that $\sigma^o(A) = \sigma(A)$ for any Abelian group having no Z_2 factors in its canonical representation. For the sake of completeness, we present here one class of Abelian groups A for which $\gamma(A) \neq \sigma(A)$.

THEOREM 9. *Let $A = Z_n \times \cdots \times Z_n$ have rank r, where $r \equiv 2 \bmod 4$, $r > 2$, and n is a prime, $n > 3$. Then $\gamma(A) < \sigma(A)$.*

Proof. White and Jungerman show that $\gamma(A) = 1 + |A|(r-2)/4$. We claim that

$$\sigma(A) \geqslant 1 + |A|(r(n-1)/n - 2) > \gamma(A).$$

Suppose A acts on the surface S. Since A has odd order the action must be orientation preserving. By Theorem 4, A has a generating set of $2g + t$

elements, where $p: S \to S/A$ has t branch points and S/A has genus g. Since each branch point must have order at least n, the Riemann–Hurwitz equation yields

$$\chi(S) \leqslant |A| \, (2 - 2g - t(n-1)/n).$$

As A has rank r, it follows that $2g + t \geqslant r$ (in fact, $2g + t \geqslant r + 1$ since the extra relation in Theorem 4 shows one generator is redundant). Therefore

$$\begin{aligned} \chi(S) &\leqslant |A| \, (2 - 2g - (r - 2g)(n-1)/n) \\ &\leqslant |A/ \, (2 - r(n-1)/n), \end{aligned}$$

which implies the claimed inequality for $\sigma(A)$. ∎

4. Space Groups and Groups of Genus 0 or 1

In this section, our viewpoint shifts to consider infinite groups acting on the plane. The groups in question are discrete groups of isometries of the Euclidean or hyperbolic plane. This means the discussion and the results of Section 1 apply here as well. For example, Proposition 1 follows immediately from the classification of Euclidean or hyperbolic isometries. This approach is more traditional and closer to the spirit of Macbeath [14], Levinson and Maskit [14], and Jones and Singerman [10].

Let Δ be a triangle lying in the Euclidean or hyperbolic plane or the unit sphere S and having angles π/p, π/q, π/r, where p, q, r are positive integers. The possible values for p, q, r are $\{3, 3, 3\}$, $\{2, 4, 4\}$, and $\{2, 3, 6\}$ if the triangle is Euclidean; $\{1, n, n\}$, $\{2, 2, n\}$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$ if the triangle is spherical; and any values such that $1/p + 1/q + 1/r < 1$ if the triangle is hyperbolic. The *full* (p, q, r)*-triangle group*, denoted $T(p, q, r)$ is the group of isometries generated by the three reflections in the sides of the triangle Δ. The study of triangle groups goes back to Fricke and Klein in the 19th Century. Magnus' book [17] is recommended for a modern account. The important facts needed here are that $T(p, q, r)$ is a discrete group of isometries having a subgroup of finite index that acts freely with compact quotient and that $T(p, q, r)$ has a presentation

$$\langle z_1, z_2, z_3 : z_1^2 = z_2^2 = z_3^2 = 1, (z_1 z_2)^p = (z_2 z_3)^q = (z_3 z_1)^r = 1 \rangle.$$

We are also interested in the index 2 subgroup of $T(p, q, r)$ consisting of orientation preserving elements. It will be denoted $T^o(p, q, r)$ and called the *ordinary* (p, q, r)*-triangle group*. It is not hard to show that $T^o(p, q, r)$ is generated by the rotations $y_1 = z_1 z_2$, $y_2 = z_2 z_3$, $y_3 = z_3 z_1 = (y_1 y_2)^{-1}$ and has a presentation

$$\langle y_1, y_2, y_3 : y_1^p = y_2^q = y_3^r = 1, y_1 y_2 y_3 = 1 \rangle.$$

A (planar) *space group* is a discrete group of isometries of the Euclidean or hyperbolic plane or the sphere having a compact quotient space. The classical Bieberbach theorems [2] state that a Euclidean space group in dimension n has a normal subgroup of finite index generated by n linearly independent translations and that the number of Euclidean space groups of a given dimension n is finite. In particular there are exactly 17 Euclidean 2-dimensional space groups. There is much literature about these "wallpaper" groups. Schattschneider's article [23] is recommended for an elementary treatment; Coxeter and Moser [5] give presentations for the groups. The three Euclidean full triangle groups $T(3, 3, 3)$, $T(2, 4, 4)$, and $T(2, 3, 6)$ are of course space groups. Moreover, every Euclidean space group is a subgroup of these full triangle groups. All spherical space groups are finite, and again all are subgroups of the full spherical triangle groups. In fact, any finite group acting on the sphere is a spherical space group and these groups have been classified (this can be derived from Theorems 4 and 5 or see [1, 13, 24]).

Every spherical space group A acts on the sphere. Thus by Theorem 4 and 5 we have

THEOREM 10. *$\gamma(A) = 0$ if A is isomorphic to a spherical space group.*

Maschke's classification [18] of groups of genus 0 is in effect the converse of this theorem: every group of genus 0 is a spherical space group.

THEOREM 11. *Let A be a finite quotient of a planar Euclidean space group. Then A acts on the torus or the sphere. In particular, $\gamma(A) \leqslant 1$.*

Proof. Suppose $A = C/N$, where C is a Euclidean space group acting on the plane S and N is a normal subgroup of finite index in C. By Proposition 2, it suffices to show that $S/C^o \cap N$ is a torus or a sphere. As we already noted, there is a normal subgroup K of finite index in C generated by two independent translations. Since N has finite index in C, the normal subgroup $K \cap N$ must also be generated by two independent translations. Clearly $S/K \cap N$ is a torus. By Proposition 2, $C^o \cap N/K \cap N$ acts on $S/K \cap N$ with quotient $S/C^o \cap N$ (notice that $K \subset C^o$ since translations preserve orientation). One trivial consequence of the Riemann–Hurwitz equation is that Euler characteristic does not decrease when taking a quotient. Therefore $S/C^o \cap N$, as a quotient of the torus $S/K \cap N$, must be a sphere or torus. ∎

Proulx's classification [20, 21] of groups of genus 1 is again almost the converse of Theorem 11: every group of genus 1 is a finite quotient of a Euclidean space group, except possibly for four special groups. The four exceptional groups of genus 1 may have presentations realizing them as quotients of Euclidean space groups but such presentations have not yet been found. Proulx obtains her classification by showing exhaustively that any group of genus 1 must have a presentation falling into one or more of various classes of presentations. She then uses the voltage graph construction to show any Cayley graph corresponding to a presentation in a given class has a toroidal embedding. Only as an afterthought in her thesis [20] does she note that the different presentation classes correspond to presentations of the Euclidean space groups (unfortunately this connection is never mentioned in the published announcement of her results [21]). Theorem 11 obviates the need for Proulx's seventeen special voltage graph constructions. However, the converse of Theorem 11, that any group of genus 1 must be a finite quotient of a Euclidean space group (with four exceptions), is still a long, difficult argument occupying nearly 100 pages of Proulx's thesis. There is no way known to the author to ease this situation at the present.

5. Hurwitz's Theorem

Hurwitz's theorem [9] bounds the order of a group A of orientation preserving conformal automorphism of a Riemann surface S of genus $g > 1$ (Schwarz had already shown that such a group must be finite). We present a version of Hurwitz's theorem that allows orientation reversing homeomorphisms and gives more specific information about groups whose order is close to the bound. The extra information is of particular interest to us because analogous conclusions can be drawn when one only has a Cayley graph for the group A embedded in the surface S rather than an action of A on S. Sequels to this paper will consider the Cayley graph version and its importance in showing there is only one group of genus 2 [27]. Define a quotient of $T(p, q, r)$ to be *proper* if the image of $T^o(p, q, r)$ has index 2, and *improper* otherwise.

THEOREM 12. *Let A act on the orientable surface S, where $\chi(S) < 0$.*

(a) *If $A = A^o$, then $|A| \leqslant 42\,|\chi(S)|$. Moreover if $|A| > 6\,|\chi(S)|$, then A is a quotient of an ordinary triangle group $T^o(p, q, r)$ and if $|A| > 12\,|\chi(S)|$ the triple (p, q, r) must be $(2, 4, 5)$ or $(2, 3, r)$, $7 \leqslant r \leqslant 11$.*

(b) *If A acts with reflections, then $|A| \leqslant 84\,|\chi(S)|$. Moreover if $|A| > 12\,|\chi(A)|$, then A is a proper quotient of a full triangle group $T(p, q, r)$ or the group $\langle y, z\colon z^2 = y^p = 1,\ [z, y]^r = 1\rangle$, where $(p, r) = (3, 4)$, $(3, 5)$, or*

(5, 2). *If* $|A| > 24\,|\chi(S)|$, *then* A *is a proper quotient of* $T(p, q, r)$, *where* (p, q, r) *is* $(2, 4, 5)$ *or* $(2, 3, r)$, $7 \leqslant r \leqslant 11$.

(c) *If* A *acts without reflections but* $A \neq A^o$, *then* $|A| \leqslant 6\,|\chi(S)|$.

Proof. The Riemann–Hurwitz equation applies to A^o whether A acts with reflections or not. Then

$$\chi(S) = |A^o| \left(\chi(S/A^o) - \sum (1 - 1/m_j)\right).$$

If $\chi(S/A^o) < 0$, then $\chi(S) \leqslant -|A^o| \leqslant -|A|/2$. If $\chi(S/A^o) = 0$, then $\chi(S) \leqslant |A^o|\,(-\frac{1}{2}) \leqslant -|A|/4$ since the number t of branch points must be nonzero, otherwise $\chi(S) = 0$. If $\chi(S/A^o) = 2$ and $t > 3$, then one easily checks that $\chi(S) \leqslant -|A^o|/6$ (four branch points, all of order two, contradicts $\chi(S) < 0$). We conclude that if $\chi(S) > -|A^o|/6$, then S/A^o is a sphere with exactly three branch points ($t = 2$ implies $\chi(S) > 0$). Let the orders of the branch points be $m_1 = p$, $m_2 = q$, $m_3 = r$. Then A^o is quotient of an ordinary triangle group $T(p, q, r)$ by Theorem 4. In all further dicussion S/A^o is a sphere with three branch points.

Suppose $A = A^o$ and $|A| > 12\,|\chi(S)|$. An inspection of the various cases for (p, q, r) shows the only possibilities are those listed in (a). The largest value for $|A|/|\chi(S)|$ is 42 occurring for the triple $(2, 3, 7)$.

Suppose A acts with reflections. Let C be the fixed circle of the action of A/A^o on S/A^o. If all three branch points of A^o lie on the circle C, then A is a proper quotient of a full triangle group $T(p, q, r)$ by Theorem 5. Again, inspection yields the possibilities for (p, q, r) listed in (b) when $|A| > 24\,|\chi(S)|$. If not all the branch points of A^o lie on C, then two must lie off C and they must have the same order. The cases already listed show this is impossible if $|A| > 24\,|\chi(S)|$. Possible triples (p, p, r) for $|A| > 12\,|\chi(S)|$ are $(5, 5, 2)$, $(3, 3, 4)$, and $(3, 3, 5)$. The resulting presentation from Theorem 5 is

$$\langle y, w, z_1, z_2 : z_1^2 = z_2^2 = y^p = (z_1 z_2)^r = 1, yw = 1, z_1 w z_2 w^{-1} = 1 \ldots\rangle.$$

Then the elimination of w and z_2 from this presentation results in the presentation given in (b) with the given values for p and r.

Suppose A acts without reflections, but $A \neq A^o$. Then the Riemann–Hurwitz equation applies to A as well as A_o. Previous arguments show that $\chi(S) \leqslant -|A|/4$ unless $\chi(S/A) > 0$. Since S/A is nonorientable, the only possibility remaining is $\chi(S/A) = 1$. If $t \geqslant 3$, then $\chi(S) \leqslant |A|\,(-\frac{1}{2})$. If $t = 2$, $\chi(S) \leqslant |A|\,(-\frac{1}{6})$ occurring when $m_1 = 2$, $m_2 = 3$. If $t = 1$, $\chi(S) > 0$, a contradiction. This completes case (c) and the theorem. ∎

COROLLARY 13. *Let* A *be a finite group such that* $\sigma(A) > 1$.

(a) $|A| \leqslant 84(\sigma^o(A)-1)$. *Moreover* $|A| > 24(\sigma^o(A)-1)$ *if and only if* A *is a quotient of* $T^o(2, 3, r)$, $7 \leqslant r \leqslant 11$ *or* $T^o(2, 4, 5)$.

(b) $|A| \leqslant 168(\sigma(A)-1)$. *Moreover* $|A| > 48(\sigma(A)-1)$ *if and only if* A *is a quotient of* $T^o(2, 3, 7)$ *or* A *is a proper quotient of* $T(p, q, r)$, *where* (p, q, r) *is* $(2, 3, r)$, $7 \leqslant r \leqslant 11$ *or* $(2, 4, 5)$.

Proof. The "only if" parts follow immediately from Theorem 12. The "if" part of (a) comes from Theorem 6. The "if" part of (b) comes from Theorem 7. ∎

It is worth noting that by (a) the only values for $|A|/(\sigma^o(A)-1)$ greater than 24 are 84, 48, 36, 30, 132/5, and 40. By (b) the only values for $|A|/(\sigma(A)-1)$ greater than 48 are 84, 168, 96, 72, 60, 264/5, and 80. In both cases the upper bound is achieved by quotients of a (2, 3, 7) group. It might appear that $T(2, 3, 7)$ has few finite quotients. In fact, the opposite is true. Conder has shown recently [4] that the full symmetric group Σ_n is a proper quotient of $T(2, 3, 7)$ for all $n > 167$ and many other $n < 167$. (G. Higman had announced previously without proof that the alternating group A_n was a quotient of $T^o(2, 3, 7)$ for sufficiently large n.)

COROLLARY 14. $\sigma(\Sigma_n) = n!/168 + 1$, *and* $\sigma^o(A_n) = \sigma(A_n) = n!/168 + 1$ *for all* $n > 167$.

Proof. The value for $\sigma(\Sigma_n)$ follows directly from Conder's result and Corollary 13(b). Since A_n is the only index 2 subgroup of Σ_n, it must form the orientation preserving part of the action of Σ_n on the surface of genus $n!/168 + 1$. It follows from Corollary 13(a) that $\sigma^o(A_n) = n!/168 + 1$. Because A_n has no index 2 subgroup, $\sigma(A_n) = \sigma^o(A_n)$. ∎

6. NONORIENTABLE SURFACES

Conder shows in [4] that the alternating group A_n, $n > 167$ is a quotient of the full triangle group $T(2, 3, 7)$ as well. Let B_n be a normal subgroup of $T(2, 3, 7)$ such that $A_n = T(2, 3, 7)/B_n$. It can be shown that any reflection in $T(2, 3, 7)$ is conjugate to one of the three reflections generating $T(2, 3, 7)$. It follows that B_n contains no reflections, as otherwise A_n would be generated by two involutions. Therefore B_n acts pseudo-freely on the hyperbolic plane H and the quotient $S = H/B_n$ is a closed surface. By Proposition 2, A_n acts on S. However, the surface S must be nonorientable because A_n contains no index 2 subgroup, implying $B_n \not\subset T^o(2, 3, 7)$. It is natural to ask now whether the results of the previous sections hold for group actions on nonorientable surfaces. In particular, if a group A acts on the nonorientable surface S must there be a Cayley graph for A embedded in S?

We use a traditional tool for handling nonorientability, the orientable double covering. If S is a nonorientable surface, the homotopy classes of orientation preserving loops at a basepoint form a subgroup of index 2 in the fundamental group of S. The unique two-sheeted covering $p: \tilde{S} \to S$ coresponding to this subgroup is the *orientable double covering* (unbranched) of S. The single nontrivial covering transformation t is a fixed-point free, orientation reversing involution of the orientable surface $\tilde{S}$. It follows from elementary facts about covering spaces (see, for example, [19]) that any homeomorphism $f: S \to S$ lifts to a homeomorphism $\tilde{f}: \tilde{S} \to \tilde{S}$ such that $fp = p\tilde{f}$. Moreover, there are only two possible lifts of f; if one is called $\tilde{f}$, the other must be $\tilde{f}t$. Since $t\tilde{f}$ is also a lift of f ($pt\tilde{f} = p\tilde{f}$), we have $t\tilde{f} = \tilde{f}t$, that is, t commutes with all lifts of homeomorphisms of S. Finally, since t is orientation reversing, exactly one of $\tilde{f}$ and $\tilde{f}t$ is orientation preserving.

Suppose the group A acts on the nonorientable surface S. Let $\tilde{a}$ stand for the orientation preserving lift of $a \in A$ to the orientable double covering $\tilde{S}$. Then for any $a, b \in A$, $p\tilde{a}\tilde{b} = ap\tilde{b} = abp$. Therefore $\tilde{a}\tilde{b}$ is a lift of ab. Since $\tilde{a}\tilde{b}$ is orientation preserving, this lift must be the one we have denoted $\widetilde{ab}$. Hence the action of A lifts to an orientation preserving action of A on $\tilde{S}$. In fact, since the covering transformation t of $p: \tilde{S} \to S$ commutes with each lift, we have an action of $Z_2 \times A$ on $\tilde{S}$, where the Z_2 factor is generated by t. We summarize this discussion as follows:

THEOREM 15. *Let S be a nonorientable surface and $p: \tilde{S} \to S$ the orientable double covering of S. Let t be the orientation reversing covering transformation for p. Let A be a group acting on S. Then $Z_2 \times A$ acts on $\tilde{S}$, where the Z_2 factor is generated by t and the A factor is orientation preserving. The action of $Z_2 \times A/Z_2$ on $\tilde{S} = S/Z_2$ is the original action of A on S. There is a Cayley graph for A embedded in S.*

Proof. To show there is a Cayley graph for A embedded in S, simply observe that by Theorem 5 there is a Cayley graph G for $Z_2 \times A$ embedded in $\tilde{S}$ invariant under the action of $Z_2 \times A$ on $\tilde{S}$, and that A acts transitively without fixed vertices on the quotient graph G/Z_2 embedded in S. ∎

Other results from the previous sections now generalize easily. For example, if A acts on the nonorientable surface S, then by the orientable Hurwitz theorem $|Z_2 \times A| \leqslant 84\,|\chi(\tilde{S})|$ so $|A| \leqslant 84\,|\chi(S)|$, since $\chi(\tilde{S}) = 2\chi(S)$. Moreover, if $|A| > 12\,|\chi(S)|$ then A must be as described in Theorem 12(b), because the action of $Z_2 \times A$ on $\tilde{S}$ falls under case (b) of Theorem 12 and A is a quotient of $Z_2 \times A$. However, since the A factor of $Z_2 \times A$ is $(Z_2 \times A)^o$, it follows that the quotient of a triangle group obtained for A is improper.

Singerman [25] has obtained similar results for nonorientable surfaces. He concentrates on finite group actions of maximal order, namely improper quotients of $T(2, 3, 7)$, which he calls H^*-groups. He shows that $PSL(2, 8)$ is an H^*-group and that there are infinitely many H^*-groups. Conder's result implies that A_n is an H^*-group for all $n > 167$.

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